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Bounds for the Norm of Certain Spline Projections*

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1. INTRODUCTION AND NOTATION

Let n and q be given natural numbers such that $n + 1 \geqslant q > 0$ $(n > 0)$. Further, let $I = [0, 1]$, and let Δ denote an arbitrary but fixed partition of the interval I: $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$. By $\mathrm{Sp}(2q - 1, \Delta)$ we denote the space of *natural spline functions of degree* $2q - 1$; thus $s \in \mathrm{Sp}(2q - 1, \Delta)$ iff:

(i) in each interval $[x_{i-1}, x_i]$ $(i = 1, 2,..., n)$ s coincides with an algebraic polynomial of degree at most $2q - 1$,

(ii) $s \in C^{2q-2}[0, 1]$,

(iii) $s^{(j)}(0) = s^{(j)}(1) = 0$ $(j = q, q + 1,..., 2q - 2)$.

It is known (see, e.g., [1]) that for given real numbers f_i $(i = 0, 1,..., n)$ there exists exactly one $s \in \mathrm{Sp}(2q - 1, \Delta)$ interpolating the data f_i:

$$s(x_i) = f_i \qquad (i = 0, 1,..., n) \tag{1.1}$$

(we may assume that $f_i = f(x_i)$, where $f \in C[0, 1]$). Every such spline function may be written in the following way:

$$s(x) = \sum_{i=0}^{n} f_i s_i(x) \qquad (x \in I),$$

where $s_i \in \mathrm{Sp}(2q - 1, \Delta)$, $s_i(x_j) = \delta_{ij}$ $(i, j = 0, 1,..., n)$. The functions s_i are the so-called *fundamental spline functions*. Consider the operator L_n^{2q-1} defined by

$$L_n^{2q-1}f(x) = \sum_{i=0}^{n} f(x_i)\, s_i(x) \qquad (f \in C[0, 1]). \tag{1.2}$$

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It is obvious that L_n^{2q-1} is a linear, bounded, and idempotent operator with domain $C[0, 1]$ and range $\mathrm{Sp}(2q - 1, \Delta)$; thus L_n^{2q-1} is a *projection*. We have the elementary but important inequality

$$\|f - L_n^{2q-1}f\|_\infty \leqslant (1 + \| L_n^{2q-1} \|) \operatorname{dist}(f, \mathrm{Sp}(2q - 1, \Delta)),$$

where $\| \cdot \|_\infty$ stands for the sup-norm in the interval I, and

$$\| L_n^{2q-1} \| = \sup_{\|f\|_\infty \leqslant 1} \| L_n^{2q-1}f \|_\infty \qquad (f \in C[0, 1]).$$

From the above inequality it follows that the information on the size of the norm of the projection L_n^{2q-1} is important. Some results on the norm of the above projection are known in the periodic case, i.e., when conditions (iii) are changed by the following $s^{(j)}0) = s^{(j)}(1)$ $(j = 0, 1,..., 2q - 2)$, but the function f in (1.2) is a periodic function such that $f(0) = f(1)$ (see [5], [10–11], [13–17]).

In Section 2 some lemmas are given. In Section 3 the cubic case ($q = 2$) is investigated. Estimations from above for $\| L_n^3 \|$ (for arbitrary knots), and from below for $\| L_n^3 \|$ (for equidistant knots) are given. In the final section a theorem is given in which the quantity $\| L_n^5 \|$ is estimated from above (in the case of equidistant knots).

2. Some Lemmas

We define the sequence $\{d_i\}$ in the following way: $d_{-1} = 0$, $d_0 = 1$, $d_{i+1} = 4d_i - d_{i-1}$ $(i = 0, 1,...)$.

Lemma 2.1. *For the numbers d_i defined as above the following inequalities hold*:

$$(2 + 3^{1/2})\, d_i < d_{i+1} \leqslant 4d_i \qquad (i = 0, 1,...). \tag{2.1}$$

Proof. Solving the above difference equation we obtain

$$d_i = [(3 - 2(3)^{1/2})(2 - 3^{1/2})^i + (3 + 2(3)^{1/2})(2 + 3^{1/2})^i]/6 \equiv a_i + b_i\,,$$

where $a_i = (3 - 2(3)^{1/2})(2 - 3^{1/2})^i/6$. Hence

$$\begin{aligned} d_{i+1} &= a_i(2 - 3^{1/2}) + b_i(2 + 3^{1/2}) = 2d_i + 3^{1/2}(d_i - 2a_i) \\ &= (2 + 3^{1/2})\, d_i - 2(3)^{1/2}\, a_i > (2 + 3^{1/2}d_i), \end{aligned}$$

since $a_i < 0$. The second inequality in (2.1) is obvious. ∎

Let $\beta_{j,-1} = \beta_{j0} = \beta_{jn} = \beta_{j,n+1} = 0$, and

$$\begin{aligned} \beta_{ij} &= (-1)^{j+i}\, d_{j-1} d_{n-i-1}/d_{n-1} \qquad (j \leqslant i), \\ &= (-1)^{j+i}\, d_{i-1} d_{n-j-1}/d_{n-1} \qquad (j \geqslant i), \\ &\qquad (i, j = 1, 2, \ldots, n-1). \end{aligned} \tag{2.2}$$

LEMMA 2.2. *If the numbers $m_j^{(i)}$ are such that*

$$\begin{aligned} &m_{j-1}^{(i)} + 4m_j^{(i)} + m_{j+1}^{(i)} = 6n^2(\delta_{j+1,i} - \delta_{ji} + \delta_{j-1,i}), \\ &m_0^{(i)} = m_n^{(i)} = 0 \qquad (i = 0, 1, \ldots, n;\ j = 1, 2, \ldots, n-1), \end{aligned} \tag{2.3}$$

then

$$\begin{aligned} m_j^{(i)} &= (-1)^{j+1}\, 6n^2 d_{n-j-1}/d_{n-1} \qquad (i = 0), \\ &= (-1)^{j+i+1}\, 36n^2 d_{j-1} d_{n-i-1}/d_{n-1} \qquad (j < i), \\ &= -6n^2(d_{i-2} d_{n-i-1} + 2d_{i-1} d_{n-i-1} + d_{i-1} d_{n-i-2})/d_{n-1} \\ &\qquad (j = i;\ i = 1, 2, \ldots, n-1), \\ &= (-1)^{j+i+1}\, 36n^2 d_{i-1} d_{n-j-1}/d_{n-1} \qquad (j > i), \\ &= m_{n-j}^{(i)} \qquad (i = n). \end{aligned} \tag{2.4}$$

Proof. It is known (see, e.g., [12]) that a matrix of the above system of linear equations (2.3) possesses an inverse matrix with entries given by (2.2). Hence, and from (2.3), we obtain

$$m_j^{(i)} = 6n^2(\beta_{j,i-1} - \beta_{ji} + \beta_{j,i+1})$$

and further, in virtue of (2.2), we obtain (2.4). ∎

LEMMA 2.3. *Let the knots x_i be equidistant* $(x_i = i/n;\ i = 0, 1, \ldots, n)$. *If $q = 2$ and $x \in [x_{j-1}, x_j]$ $(j = 1, 2, \ldots, n)$ then*

$$\begin{aligned} \operatorname{sgn} s_i(x) &= (-1)^{i+j} \qquad (j \leqslant i), \\ &= (-1)^{i+j+1} \qquad (j > i), \\ &\qquad (i = 0, 1, \ldots, n;\ j = 1, 2, \ldots, n). \end{aligned} \tag{2.5}$$

Proof. If $x \in [x_{j-1}, x_j]$ then the fundamental cubic spline function $s_i(x)$ may be written in the following way:

$$\begin{aligned} s_i(x) = {}& \delta_{j-1,i}(1-t) + \delta_{ij} t \\ &+ \{m_{j-1}^{(i)}[(1-t)^3 - (1-t)] + m_j^{(i)}(t^3 - t)\}/6n^2, \end{aligned} \tag{2.6}$$

where $t = n(x - x_{j-1})$. The proof of the above equality in (2.5) will be divided into three cases.

Case 1°. $|i-j| > 1$. Let $i = 0$. From (2.6) and (2.4) we have

$$s_0(x) = (-1)^j[(t-2)\, d_{n-j} + (t+1)\, d_{n-j-1}]\, t(1-t)/d_{n-1}.$$

By virtue of Lemma 2.1 it follows that the expression in the square brackets is negative for $0 \leqslant t \leqslant 1$. Hence $\operatorname{sgn} s_0(x) = (-1)^{j+1}$ for $x \in [x_{j-1}, x_j]$. Quite similarly we can prove (2.5) for $i > 0$.

Case 2°. $j = i + 1$ $(i = 0, 1,..., n-1)$. By virtue of (2.6) we have the following expression for the fundamental spline function $s_i(x)$ $(x \in [x_{j-1}, x_j])$:

$$s_i(x) = (1-t)\{1 + [m_i^{(i)}(t^2 - 2t) - m_{i+1}^{(i)}(t^2 + t)]/6n^2\}.$$

From (2.4) it follows that $m_i^{(i)} < 0$, $m_{i+1}^{(i)} > 0$. Hence a coefficient before t^3 in the last expression is a positive. For $i > 0$ $s_i(x)$ vanish in x_{i-1}, x_{i+1}, and on the right of x_{i+1}. Thus $s_i(x) > 0$ for $x \in [x_i, x_{i+1})$.

Case 3°. $j = i$ $(i = 1, 2,..., n)$. In this case we have

$$s_i(x) = t\{1 + [m_{i-1}^{(i)}(-t^2 + 3t - 2) + m_i^{(i)}(t^2 - 1)]/6n^2\}.$$

Let $i = 2, 3,..., n-1$. From (2.4) we obain $m_{i-1}^{(i)} > 0$, $m_i^{(i)} < 0$. Hence $s_i(x)$ vanish in the points x_{i+1}, x_{i-1} and on the left of x_{i-1}. Finally $s_i(x) > 0$ if $x \in (x_{i-1}, x_i]$. Similarly we can prove that $s_1(x) > 0$ if $x \in (x_0, x_1]$ and $s_n(x) > 0$ if $x \in (x_{n-1}, x_n]$. ∎

3. Cubic Case

Now we introduce some additional notation. Let $h_j = x_j - x_{j-1}$ $(j = 1, 2,..., n)$, $h = \max_{1 \leqslant j \leqslant n} h_j$, $\mathbf{h} = \min_{1 \leqslant j \leqslant n} h_j$, $M_n = h/\mathbf{h}$, $\lambda_j = h_{j+1}/(h_j + h_{j+1})$, $\mu_j = 1 - \lambda_j$ $(j = 1, 2,..., n-1)$, $m_j = s''(x_j)$ $(j = 0, 1,..., n)$, where $s \in \mathrm{Sp}(3, \Delta)$.

The following theorem holds

THEOREM 3.1. *For arbitrary knots* x_i $(i = 0, 1,..., n)$,

$$\| L_n^3 \| \leqslant 1 + \tfrac{3}{2} M_n^2.$$

Proof. The above defined numbers m_j satisfied the so-called *consistency relations* (see, e.g., [1])

$$\mu_j m_{j-1} + 2m_j + \lambda_j m_{j+1} = \frac{6}{h_j + h_{j+1}} \left(\frac{f_{j+1} - f_j}{h_{j+1}} - \frac{f_j - f_{j-1}}{h_j} \right)$$

$$(j = 1, 2, \ldots, n-1;\ m_0 = m_n = 0). \quad (3.1)$$

Using a standard diagonal dominance argument to the above system (3.1) we obtain

$$\max_{1 \leqslant i \leqslant n-1} |m_j| \leqslant 6\omega(f, h)/\mathbf{h}^2, \quad (3.2)$$

where $\omega(\cdot, \cdot)$ denotes usual mudulus of continuity. For $x \in [x_{j-1}, x_j]$ the spline function $s(x)$ has the form

$$s(x) = f_{j-1}(1 - t) + f_j t + \frac{h_j^2}{6} \{m_{j-1}[(1 - t)^3 - (1 - t)] + m_j(t^3 - t)\},$$

where $t = (x - x_{j-1})/h_j$. Hence, and from (3.2), we obtain

$$|s(x)| \leqslant \|f\|_\infty + \tfrac{3}{4} M_n{}^2 \omega(f, h). \quad (3.3)$$

For the function $f \in C[0, 1]$, and such that $\|f\|_\infty \leqslant 1$, the obvious inequality $\omega(f, h) \leqslant 2$ holds. Hence, and from (3.3), we obtain the desired inequality in the thesis of the above theorem. ∎

COROLLARY 3.1. *For equidistant knots we have* $\| L_n{}^3 \| \leqslant \frac{5}{2}$.

Now some estimations from below for $\| L_n{}^3 \|$ will be given in the case of equidistant knots. Let

$$\Lambda_n^{2q-1}(x) = \sum_{l=0}^{n} |s_l(x)| \qquad (x \in I)$$

denote the so-called *Lebesgue function* for the projection L_n^{2q-1}. It is known that $\| L_n^{2q-1} \| = \| \Lambda_n^{2q-1} \|_\infty$. Now we give the explicit form for the function $\Lambda_n{}^3(x)$ $(x \in I)$ in the case when knots x_i are equidistant. By virtue of (2.6) and (2.5) we have for $x \in [x_{i-1}, x_i]$,

$$\Lambda_n{}^3(x) = \sum_{l=0}^{n} |s_l(x)| = \sum_{l=0}^{i-1} (-1)^{i+l+1} s_l(x) + \sum_{l=i}^{n} (-1)^{i+l} s_l(x)$$

$$= 1 + \sum_{l=0}^{i-1} (-1)^{i+l+1} [m_{i-1}^{(l)} C_i(x) + m_i^{(l)} D_i(x)]$$

$$+ \sum_{l=i}^{n} (-1)^{i+l} [m_{i-1}^{(l)} C_i(x) + m_i^{(l)} D_i(x)].$$

where

$$C_i(x) = [(1-t)^3 - (1-t)]/6n^2, \qquad D_i(x) = (t^3 - t)/6n^2$$
$$(t = n(x - x_{i-1});\ i = 1, 2,\dots, n). \quad (3.4)$$

For $x \in [x_{i-1}, x_i]$ we have $C_i(x) \leqslant 0$, $D_i(x) \leqslant 0$. Let

$$\begin{aligned} \alpha_{i,n} &= \sum_{l=0}^{i-1} (-1)^{i+l+1}\, m_{i-1}^{(l)} + \sum_{l=i}^{n} (-1)^{i+l}\, m_{i-1}^{(l)}, \\ \beta_{i,n} &= \sum_{l=0}^{i-1} (-1)^{i+l+1}\, m_i^{(l)} + \sum_{l=i}^{n} (-1)^{i+l}\, m_i^{(l)}. \end{aligned} \quad (3.5)$$

The Lebesgue function $\Lambda_n^3(x)$ may be written in terms $\alpha_{i,n}$ and $\beta_{i,n}$ in the following way:

$$\Lambda_n^3(x) = 1 + \alpha_{i,n} C_i(x) + \beta_{i,n} D_i(x) \qquad (x \in [x_{i-1}, x_i];\ i = 1, 2,\dots, n). \quad (3.6)$$

The numbers $\alpha_{i,n}$ and $\beta_{i,n}$ may be expressed by the numbers d_k. Thus by virtue of (2.4) we have

$$\begin{aligned} \alpha_{i,n} &= -\frac{6n^2}{d_{n-1}} \left[\left(1 + 6 \sum_{l=0}^{i-3} d_l\right) d_{n-i} + d_{i-3} d_{n-i} + 2 d_{i-2} d_{n-i} + d_{i-2} d_{n-i-1} \right. \\ &\qquad \left. - \left(1 + 6 \sum_{l=0}^{n-i-1} d_l\right) d_{i-2}\right], \\ \beta_{i,n} &= -\frac{6n^2}{d_{n-1}} \left[\left(1 + 6 \sum_{l=0}^{n-i-2} d_l\right) d_{i-1} + d_{i-2} d_{n-i-1} + 2 d_{i-1} d_{n-i-1} \right. \\ &\qquad \left. + d_{i-1} d_{n-i-2} - \left(1 + 6 \sum_{l=0}^{i-2} d_l\right) d_{n-i-1}\right] \qquad (i = 1, 2,\dots, n). \end{aligned} \quad (3.7)$$

THEOREM 3.2. *Let $x_i = i/n$ $(i = 0, 1,\dots, n)$. Then*

$$\begin{aligned} \| L_n^3 \| &\geqslant \gamma_n \quad \text{for } n = 2m + 1, \\ &\geqslant \delta_n \quad \text{for } n = 2m, \end{aligned} \qquad (m = 1, 2,\dots),$$

where

$$\gamma_n = 1 + \frac{3}{4 d_{n-1}} (d_{2j-1+k} - d_{2j-2+k}) \left(6 \sum_{l=0}^{2j-3-k} d_l + d_{2j-2+k} + d_{2j-3+k} + 1\right),$$

$$\begin{aligned} k &= 0 \quad \text{for } n = 4j - 1 \quad (j = 1, 2,\dots), \\ &= -1 \quad \text{for } n = 4j - 3 \quad (j = 2, 3,\dots), \end{aligned}$$

$$\delta_n = 1 + \frac{3}{8d_{n-1}} \times \left[(d_{2j+k} - d_{2j-2+k})\left(1 + 6\sum_{l=0}^{2j-2+k} d_l\right) - 2d_{2j-1+k}(d_{2j-1+k} + d_{2j-2+k})\right],$$

$$\begin{aligned} k &= 0 \quad \text{for} \quad n = 4j, \\ &= -1 \quad \text{for} \quad n = 4j-2. \end{aligned} \qquad (j = 1, 2,\ldots).$$

Additionally $\| L_1^3 \| = 1$.

Proof. From (3.4) and (3.6)–(3.7) it follows that $\Lambda_n^3(x) = \Lambda_n^3(1-x)$ $(x \in I)$. Thus investigation of the function $\Lambda_n^3(x)$ may be done only for $x \in [0, 1/2]$.

Assume n is odd. Let

1°. $n = 4j - 1$ $(j = 1, 2,\ldots)$. Then putting $i = 2j$ in (3.6) and (3.7), we obtain

$$\alpha_{2j,4j-1} = -\frac{6n^2}{d_{n-1}}\left[\left(1 + 6\sum_{l=0}^{2j-3} d_l\right) d_{2j-1} + d_{2j-3}d_{2j-1} + 2d_{2j-2}d_{2j-1} + d_{2j-2}^2 - \left(1 + 6\sum_{l=0}^{2j-3} d_l\right) d_{2j-2}\right].$$

Using the obvious equality $d_{2j-3}d_{2j-1} + 2d_{2j-2}d_{2j-1} - 5d_{2j-2}^2 = (d_{2j-1} - d_{2j-2})(d_{2j-2} + d_{2j-3})$ we obtain finally

$$\alpha_{2j,4j-1} = -\frac{6n^2}{d_{n-1}}(d_{2j-1} - d_{2j-2})\left(6\sum_{l=0}^{2j-3} d_l + d_{2j-2} + d_{2j-3} + 1\right) < 0,$$

$$\beta_{2j,4j-1} = \alpha_{2j,4j-1}\,. \tag{3.8}$$

From (3.6), (3.4), and (3.8) it follows that the function $\Lambda_n^3(x)$ is strictly concave in the interval (x_{2j-1}, x_{2j}), and hence $\max_{x_{2j-1} \leqslant x \leqslant x_{2j}} \Lambda_n^3(x) = \Lambda_n^3(1/2) = \gamma_n$.

2°. $n = 4j - 3$ $(j = 2, 3,\ldots)$. In this case we put $i = 2j - 1$. Similarly calculations as above give the desired result. For $n = 1$ from (2.3) and (3.5)–(3.6) it follows that $\Lambda_1^3(x) \equiv 1$. Hence $\| L_1^3 \| = 1$.

Assume n is even. Let

3°. $n = 4j - 2$ $(j = 1, 2,\ldots)$. Putting $i = 2j - 1$ in (3.6) we obtain by virtue of (3.7)

$$\alpha_{2j-1,4j-2} = -\frac{6n^2}{d_{n-1}}(d_{2j-1} - d_{2j-3})\left(1 + 3\sum_{l=0}^{2j-3} d_l\right) - 2\beta_{2j-1,4j-2},$$

$$\beta_{2j-1,4j-2} = -\frac{12n^2}{d_{n-1}} d_{2j-2}(d_{2j-2} + d_{2j-3}) < 0.$$

Now we can prove that $\alpha_{2j-1,4j-2} \leqslant 0$. The equivalent inequality to the above is the following:

$$(d_{2j-1} - d_{2j-3})\left(1 + 6\sum_{l=0}^{2j-3} d_l\right) \geqslant 4d_{2j-2}(d_{2j-2} + d_{2j-3}).$$

Let L denote the left hand of the above inequality. Further we have

$$L = 2(2d_{2j-2} - d_{2j-3})\left(1 + 6\sum_{l=0}^{2j-3} d_l\right) > 12d_{2j-3}\left(1 + 6\sum_{l=0}^{2j-3} d_l\right).$$

The last inequality follows from the inequality $d_{2j-2} > 3.5d_{2j-3}$ (see Lemma 2.1). Further, by virtue of $4d_{i-1} \geqslant d_i$, we obtain

$$L > 3d_{2j-2}\left(1 + 6\sum_{l=0}^{2j-3} d_l\right) = 4d_{2j-2}\left(.75 + 4.5\sum_{l=0}^{2j-3} d_l\right)$$
$$> 4d_{2j-2}(d_{2j-3} + d_{2j-2}).$$

Thus the function $\Lambda_n^3(x)$ is strictly concave in the interval (x_{2j-2}, x_{2j-1}). Putting $\delta_n = \Lambda_n^3(1/2(x_{2j-2} + x_{2j-1}))$ we obtain the desired result.

4°. $n = 4j$ $(j = 1, 2,\ldots)$. In this case we take $i = 2j$, and define $\delta_n = \Lambda_n^3(1/2(x_{2j-1} + x_{2j}))$. ∎

Now we give some numerical values for the quantities γ_n and δ_n for small values of n:

$$\gamma_3 = 1\frac{3}{10} = 1.3, \qquad \gamma_5 = 1\frac{9}{19} = 1.4736\ldots,$$

$$\gamma_7 = 1\frac{75}{142} = 1.5281\ldots, \qquad \gamma_9 = 1\frac{2448}{4505} = 1.5433\ldots,$$

$$\delta_2 = 1\frac{3}{16} = 1.1875, \qquad \delta_4 = 1\frac{29}{68} = 1.3883\ldots,$$

$$\delta_6 = 1\frac{521}{1040} = 1.5009\ldots, \qquad \delta_8 = 1\frac{23283}{43546} = 1.5357\ldots.$$

Conjecture. For all odd n $(n > 3)$ $\gamma_n = \| L_n^3 \|$. For all natural n $(n > 0)$ $\| L_n^3 \| < (1 + 3(3^{1/2}))/4 = 1.5490\ldots$.

4. Quintic Case

Now we assume that the knots x_i are equidistant. Let $s_i^{(j)} = s^{(j)}(x_i)$ $(i = 0, 1,..., n;\ j = 0, 1, 2, 3, 4)$, where $s \in \mathrm{Sp}(5, \Delta)$. First we prove the following

LEMMA 4.1. *For the equidistant knots x_i the following estimations hold:*

$$\max_{0 \leqslant i \leqslant n} |s_i'| \leqslant \frac{23}{3} n\omega\left(f, \frac{1}{n}\right), \qquad \max_{0 \leqslant i \leqslant n} |s_i''| \leqslant \frac{34}{3} n^2\omega\left(f, \frac{1}{n}\right),$$

$$\max_{0 \leqslant i \leqslant n} |s_i'''| \leqslant 40n^3\omega\left(f, \frac{1}{n}\right), \qquad \max_{0 \leqslant i \leqslant n} |s_i^{\mathrm{IV}}| \leqslant 80n^4\omega\left(f, \frac{1}{n}\right).$$

Proof. Let $A = (a_{ij})$ be a symmetric and five-diagonal matrix $(n-2) \times (n-2)$ and such that $a_{ii} = 66$ $(i = 1, 2,..., n-2)$, $a_{i,i+1} = 26$ $(i = 1, 2,..., n-3)$, $a_{i,i+2} = 1$ $(i = 1, 2,..., n-4)$, $a_{ij} = 0$ for $|i-j| > 2$ $(i, j = 1, 2,..., n-2)$. Further, let the numbers γ_j $(j = 1, 2,..., n-2)$ be the solution of the following system of linear equations:

$$\sum_{j=1}^{n-2} a_{ij}\gamma_j = 2n^3\Delta^3 f_{i-1} \qquad (i = 1, 2,..., n-2), \tag{4.1}$$

where $f_i = s(x_i)$ $(i = 0, 1,..., n)$. Using the standard diagonal dominance argument we obtain $\| A^{-1} \|_\infty = 1/12$ (here $\| \cdot \|_\infty$ stands for the infinity norm of the square matrix). Further, the obvious inequality $|2n^3\Delta^3 f_{i-1}| \leqslant 8n^3\omega(f, 1/n)$ holds. From the two above inequalities we obtain

$$\max_{1 \leqslant i \leqslant n-2} |\gamma_i| \leqslant \frac{2}{3} n^3\omega\left(f, \frac{1}{n}\right). \tag{4.2}$$

Some simple connections between the quantities γ_i and $s_i^{(j)}$ were given by Herriot and Reinsch [8], namely,

$$s_i^{\mathrm{IV}} = 60n(\gamma_i - \gamma_{i-1}) \qquad (i = 1, 2,..., n-1), \tag{4.3}$$

$$s_i''' = 30(\gamma_i + \gamma_{i-1}) \qquad (i = 1, 2,..., n-1), \tag{4.4}$$

$$s_i'' = n^2\Delta^2 f_{i-1} + \frac{1}{2n}(\gamma_{i-2} + 7\gamma_{i-1} - 7\gamma_i - \gamma_{i+1}) \qquad (i = 1, 2,..., n-1),$$

$$s_0'' = n^2\Delta^2 f_0 - \frac{1}{2n}(27\gamma_1 + \gamma_2), \tag{4.5}$$

$$s_n'' = n^2\Delta^2 f_{n-2} + \frac{1}{2n}(\gamma_{n-3} + 27\gamma_{n-2}),$$

$$s_i' = \frac{n}{2}(f_{i+1} - f_{i-1}) - \frac{1}{4n^2}(\gamma_{i-2} + 19\gamma_{i-1} + 19\gamma_i + \gamma_{i+1}) \quad (i = 1, 2,..., n-1),$$

$$s_0' = n\Delta f_0 - \frac{n}{2}\Delta^2 f_0 + \frac{1}{4n^2}(25\gamma_1 + \gamma_2), \tag{4.6}$$

$$s_n' = n\Delta f_{n-1} + \frac{n}{2}\Delta^2 f_{n-2} - \frac{1}{4n^2}(\gamma_{n-3} + 25\gamma_{n-2})$$

(we assume here that $\gamma_{-1} = \gamma_0 = \gamma_{n-1} = \gamma_n = 0$). From relations (4.2)–(4.6) the desired inequalities of this lemma follow. ∎

THEOREM 4.1. *For equidistant knots we have* $\| L_n^5 \| = 21/4$.

The proof (in which Lemma 4.1 is used) is quite similar to the that of [17, Theorem 2]. For this reason it is omitted.

REFERENCES

1. J. H. AHLBERG, E. N. NILSON, AND J. L. WALSH, "The Theory of Splines and Their Applications," Academic Press, New York, 1967.
2. C. DE BOOR, "The Method of Projections as Applied to the Numerical Solution of Two Point Boundary Value Problems Using Cubic Splines," Ph.D. dissertation, University of Michigan, Ann Arbor, 1966.
3. C. DE BOOR, On bounding spline interpolation, *J. Approximation Theory* **14** (1975), 191–203.
4. C. DE BOOR, On cubic spline functions that vanish at all knots, *Advances in Math.* **20** (1976), 1–17.
5. E. W. CHENEY AND F. SCHURER, A note on the operators arising in spline approximation, *J. Approximation Theory* **1** (1968), 94–102.
6. C. A. HALL, Natural cubic and bicubic spline interpolation, *SIAM J. Numer. Anal.* **10** (1973), 1055–1060.
7. C. A. HALL AND W. W. MEYER, Optimal error bounds for cubic spline interpolation, *J. Approximation Theory* **16** (1976), 105–122.
8. J. G. HERRIOT AND CH. H. REINSCH, "Algol 60 Procedures for the Calculation of Interpolating Natural Quintic Spline Functions," Report STAN-CS-74-402, Computer Science Department, Stanford University, January 1974.
9. D. KERSHAW, A note on the convergence of interpolatory cubic splines, *SIAM J. Numer. Anal.* **8** (1971), 67–74.
10. M. MARSDEN, Cubic spline interpolation of continuous functions, *J. Approximation Theory* **10** (1974), 103–111.
11. G. MEINARDUS, Über die norm des operators der Kardinalen spline-interpolation, *J. Approximation Theory* **16** (1976), 289–298.
12. E. NEUMAN, On inverting some band matrices (in Polish), *Matematyka Stosowana* **9** (1977), 15–24.
13. F. B. RICHARDS, Best bounds for the uniform periodic spline interpolation operator, *J. Approximation Theory* **7** (1973), 302–317.

14. F. B. RICHARDS, The Lebesgue constants for cardinal spline interpolation, *J. Approximation Theory* **14** (1975), 83–92.
15. F. SCHURER AND E. W. CHENEY, On interpolating cubic splines with equally-spaced nodes, *Nederl. Akad. Wetensch. Proc. Ser. A* **71** (1968), 517–524.
16. F. SCHURER, A note on interpolating periodic quintic splines with equally spaced nodes, *J. Approximation Theory* **1** (1968), 493–500.
17. F. SCHURER, A note on interpolating periodic quintic spline functions, *in* "Approximation Theory" (A. Talbot, Ed.), pp. 71–81, Academic Press, London, 1970.